Problems

Problem 2.1 Determine the domain of the following functions:

(i) $f(x) = \dfrac{1}{x^2-5x+6}$;

(ii) $f(x) = \sqrt{1-x^2} + \sqrt{x^2-1}$;

(iii) $f(x) = \dfrac{1}{x-\sqrt{1-x^2}}$;

(iv) $f(x) = \sqrt{1-\sqrt{4-x^2}}$;

(v) $f(x) = \dfrac{1}{1-\log x}$;

(vi) $f(x) = \log(x-x^2)$;

(vii) $f(x) = \dfrac{\sqrt{5-x}}{\log x}$;

(viii) $f(x) = \arcsin(\log x)$.

Problem 2.2

(a) If f and g are both odd functions, what are $f+g$, fg, and $f\circ g$?

(b) And what are the same functions if now f is even and g is odd?

Problem 2.3 Check whether the following functions are even or odd:

(i) $f(x) = \dfrac{x}{x^2+1}$;

(ii) $f(x) = \dfrac{x^2-x}{x^2+1}$;

(iii) $f(x) = \dfrac{\sin x}{x}$;

(iv) $f(x) = \cos(x^3)\sin(x^2)e^{-x^4}$;

(v) $f(x) = \dfrac{1}{\sqrt{x^2+1}-x}$;

(vi) $f(x) = \log\left(\sqrt{x^2+1}-x\right)$.

Problem 2.4 For which numbers $a,b,c,d \in \mathbb{R}$ the function $f(x) = \dfrac{ax+b}{cx+d}$ is its own inverse (i.e., $f\circ f = \mathrm{Id}$) in the domain of f?

Problem 2.5 Check that the function $f(x) = \dfrac{x+3}{1+2x}$ is bijective and maps its domain $\mathbb{R}-\{-1/2\}$ to $\mathbb{R}-\{1/2\}$.

Problem 2.6

(a) Determine which of these functions are injective. For those that are obtain their inverse. For those that are not, find two points with the same image.

(i) $f(x) = 7x-4$;

(ii) $f(x) = \sin(7x-4)$;

(iii) $f(x) = (x+1)^3+2$;

(iv) $f(x) = \dfrac{x+2}{x+1}$;

(v) $f(x) = x^2-3x+2$;

(vi) $f(x) = \dfrac{x}{x^2+1}$;

(vii) $f(x) = e^{-x}$;

(viii) $f(x) = \log(x+1)$.

(b) Prove that $f(x) = x^2-3x+2$ is injective in $(3/2,\infty)$.

(c) Prove that $f(x) = \dfrac{x}{x^2+1}$ is injective in $(1,\infty)$ and find $f^{-1}(\sqrt{2}/3)$.

(d) Determine if those same functions are surjective and bijective in their domains.

Problem 2.7 Calculate:

(i) $\arctan\frac{1}{2} + \arctan\frac{1}{3}$; (ii) $\arctan 2 + \arctan 3$; (iii) $\arctan\frac{1}{2} + \arctan\frac{1}{5} + \arctan\frac{1}{8}$.

HINT: Calculate the tangent of those expressions using the formula for the tangent of the sum and paying attention to the signs.

Problem 2.8 Simplify the following expressions:

(i) $f(x) = \sin(\arccos x)$;
(ii) $f(x) = \sin(2\arcsin x)$;
(iii) $f(x) = \tan(\arccos x)$;
(iv) $f(x) = \sin(2\arctan x)$;
(v) $f(x) = \cos(2\arctan x)$;
(vi) $f(x) = e^{4\log x}$.

Problem 2.9 Solve, for $x, y > 0$, the system of equations

$$\begin{cases} x^y = y^x, \\ y = 3x. \end{cases}$$

Problem 2.10

(a) Describe the function g in terms of f in the following cases ($c \in \mathbb{R}$ is a constant):

(i) $g(x) = f(x) + c$;
(ii) $g(x) = f(x + c)$;
(iii) $g(x) = f(cx)$;
(iv) $g(x) = f(1/x)$;
(v) $g(x) = f(|x|)$;
(vi) $g(x) = |f(x)|$;
(vii) $g(x) = 1/f(x)$;
(viii) $g(x) = [f(x)]_+ \equiv \max\{f(x), 0\}$.

(b) Plot the functions when $f(x) = x^2$.
(c) Plot the functions when $f(x) = \sin x$.

Problem 2.11 Sketch, using the least possible calculations, the graph of the following functions:

(i) $f(x) = (x+2)^2 - 1$;
(ii) $f(x) = \sqrt{4 - x}$;
(iii) $f(x) = x^2 + \dfrac{1}{x}$;
(iv) $f(x) = \dfrac{1}{1+x^2}$;
(v) $f(x) = \min\{x, x^2\}$;
(vi) $f(x) = |e^x - 1|$;
(vii) $f(x) = \sqrt{|x| - x}$;
(viii) $f(x) = \dfrac{1}{\lfloor 1/x \rfloor}$;
(ix) $f(x) = |x^2 - 1|$;
(x) $f(x) = 1 - e^{-x}$;
(xi) $f(x) = \log(x^2 - 1)$;
(xii) $f(x) = x \sin(1/x)$.

HINT: In (viii) $\lfloor x \rfloor$ denotes the integer part of x, i.e., the largest integer $n \leqslant x$.

Problem 2.12

(a) Prove that $\cosh x$ is even and $\sinh x$ is odd.
(b) Prove the identities $\cosh^2 x - \sinh^2 x = 1$ and $\sinh(2x) = 2 \sinh x \cosh x$.